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Linear Lagrange equations generalizing rotation

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Abstract. An N-dimensional linear Lagrange equation is given which generalizes some classical problems. Particular interest is paid to the equation generalizing a rotation. The general solution is obtained with the use of a matrix exponential method. When applied to the three-dimensional motion of a heavy particle near the surface of the earth, the results are in agreement with known results found by other methods.

1. Introduction

Consider a linear equation of motion in N dimensions:

$$\ddot{q}+\mathbf{A}(t)\,.\,\dot{q}+\mathbf{B}(t)\,.\,q = F(t), \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two $N\times N$ matrices. On account of the Helmholtz conditions (Helmholtz 1887) the necessary and sufficient conditions for (1) to be a Lagrange equation, are

$$\mathbf{B} = \tfrac{1}{2}\dot{\mathbf{A}}+\mathbf{S}, \tag{2a}$$

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}}, \tag{2b}$$

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}, \tag{2c}$$

$\mathbf{S}$ being an arbitrary symmetric $N\times N$ matrix. The lagrangian of the system is

$$L = \tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}\dot{q}\,.\,\mathbf{A}\,.\,q-\tfrac{1}{2}q\,.\,\mathbf{S}\,.\,q+F\,.\,q. \tag{3}$$

If $\mathbf{S}\equiv 0$ and $F\equiv 0$, equation (1) under conditions (2) is the generalization of the equation of motion for a charged particle in a uniform time-dependent magnetic field, as was recently discussed in a paper by Engels and Sarlet (1973).

If $\mathbf{S}\equiv\frac{1}{4}\mathbf{A}^2$, equation (1) is, as will be discussed in § 3 of this paper, the generalization of the equation of motion for a particle in a rotating frame. $F(t)$ stands for the generalized force per unit mass and the rotation is represented by the skew-symmetric matrix $\mathbf{\Omega} = \frac{1}{2}\mathbf{A}$. The equation of motion is then

$$\ddot{q}+2\mathbf{\Omega}\,.\,\dot{q}+\dot{\mathbf{\Omega}}\,.\,q+\mathbf{\Omega}^2\,.\,q = F. \tag{4}$$

This equation can be reduced to a canonical form (no term with first derivative) with the help of a linear transformation

$$q = \mathbf{G}\,.\,u, \tag{5}$$

where $\mathbf{G}$ is any particular solution of the matrix differential equation

$$\dot{\mathbf{G}}+\mathbf{\Omega}\,.\,\mathbf{G} = 0. \tag{6}$$

Such a particular solution is given by

$$\mathbf{G} = \exp\left(-\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t\right) = \sum_{j=0}^{\infty} \frac{(-\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t)^j}{j!}, \tag{7}$$

under the condition that $\boldsymbol{\Omega}$ commutes with its integral, which is certainly the case when the rotation is unidirectional ($\boldsymbol{\Omega}(t) = \omega(t)\boldsymbol{\Omega}_0$) or constant. The canonical equation is

$$\ddot{\boldsymbol{u}} = \exp\left(\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t\right).\boldsymbol{F}, \tag{8}$$

which can immediately be solved; substitution of $\boldsymbol{u}$ in (5) yields the general solution of (4):

$$\boldsymbol{q} = \exp\left(-\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t\right).\int_0^t \mathrm{d}t \int_0^t \exp\left(\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t\right).\boldsymbol{F}\,\mathrm{d}t + \exp\left(-\int_0^t \boldsymbol{\Omega}\,\mathrm{d}t\right).(\boldsymbol{a}t+\boldsymbol{b}), \tag{9}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two integration constants.

Taking from now on $\boldsymbol{\Omega}$ and $\boldsymbol{F}$ constant, one has

$$\boldsymbol{q} = \exp(-\boldsymbol{\Omega}t).\int_0^t \mathrm{d}t \int_0^t \exp(\boldsymbol{\Omega}t)\,\mathrm{d}t.\boldsymbol{F} + \exp(-\boldsymbol{\Omega}t).(\boldsymbol{a}t+\boldsymbol{b}). \tag{10}$$

This solution, however, remains purely formal, as long as the matrix exponentials must be written out as infinite series according to the definition (7). Therefore we look for a closed expression for (10).

2. General solution in closed form

The matrix exponential $\exp(\boldsymbol{\Omega}t)$ is defined as an infinite power series in $\boldsymbol{\Omega}$. This will also be the form taken by the matrix factors $\exp(-\boldsymbol{\Omega}t)$ and $\exp(-\boldsymbol{\Omega}t).\int_0^t \mathrm{d}t\int_0^t \exp(\boldsymbol{\Omega}t)\,\mathrm{d}t$ occurring in (10) and, more generally, by any matrix operator $\mathbf{f}(\boldsymbol{\Omega})$ of $\exp(\boldsymbol{\Omega}t)$, which in addition to products, also may contain time derivations and time integrations. Since $\boldsymbol{\Omega}$ satisfies its minimal polynomial, which we suppose to be of degree n ($n \leqslant N$), it is possible to express the powers of $\boldsymbol{\Omega}$, starting from the nth power, as a linear combination of smaller powers of $\boldsymbol{\Omega}$. This leads us to write

$$\mathbf{f}(\boldsymbol{\Omega}) = \lambda_0\mathbf{I} + \lambda_1\boldsymbol{\Omega} + \ldots + \lambda_{n-1}\boldsymbol{\Omega}^{n-1}, \tag{11}$$

where the n coefficients $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ must be determined as functions of time.

As it is always possible to diagonalize $\boldsymbol{\Omega}$, use can be made of the Lagrange–Sylvester interpolation formula (Gantmacher 1959)

$$\mathbf{f}(\boldsymbol{\Omega}) = \sum_{r=1}^{n} f(\alpha_r)\left(\sum_{\substack{1\\ s\neq r}}^{n} (\boldsymbol{\Omega}-\alpha_s\mathbf{I})\right)\left(\sum_{\substack{1\\ s\neq r}}^{n} (\alpha_r-\alpha_s)\right)^{-1}, \tag{12}$$

where α_r are the n different eigenvalues of $\boldsymbol{\Omega}$. Now, since $\boldsymbol{\Omega}$ is a skew-symmetric matrix, its eigenvalues are purely imaginary and complex conjugate in pairs or zero; if n is odd, at least one eigenvalue is zero. Putting $m = \frac{1}{2}n$ if n is even, and $m = \frac{1}{2}(n-1)$ if n is odd, the different eigenvalues of $\boldsymbol{\Omega}$ will be denoted by $\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_m, -\mathrm{i}\omega_1, \ldots, -\mathrm{i}\omega_m, 0$, writing the zero only if n is odd.

The coefficients λ_j must be determined by identification of (11) and (12). For a simple calculation from the minimal polynomial, we replace in (11) and (12) $\mathbf{\Omega}$ by a variable z (and $\mathbf{I}$ by 1), and write

$$f(z) = \lambda_0+\lambda_1 z+\ldots+\lambda_{n-1}z^{n-1}$$

$$= \sum_{j=1}^{m} \frac{f(\mathrm{i}\omega_j)k(z)}{(z-\mathrm{i}\omega_j)k'(\mathrm{i}\omega_j)}+\sum_{j=1}^{m} \frac{f(-\mathrm{i}\omega_j)k(z)}{(z+\mathrm{i}\omega_j)k'(-\mathrm{i}\omega_j)}+\frac{f(0)k(z)}{zk'(0)}\delta_{n,2m+1} \tag{13}$$

$$= \sum_{j=1}^{m} \frac{f(\mathrm{i}\omega_j)+(-1)^{n+1}f(-\mathrm{i}\omega_j)}{k'(\mathrm{i}\omega_j)}\left(\frac{zk(z)}{z^2+\omega_j^2}\right)$$

$$+\sum_{j=1}^{m} \frac{\mathrm{i}\omega_j[f(\mathrm{i}\omega_j)+(-1)^n f(-\mathrm{i}\omega_j)]}{k'(\mathrm{i}\omega_j)}\left(\frac{k(z)}{z^2+\omega_j^2}\right)+\frac{f(0)}{k'(0)}\left(\frac{k(z)}{z}\right)\delta_{n,2m+1}.$$

$k(z)$ is nothing else but the minimal polynomial of $\mathbf{\Omega}$:

$$k(z) \equiv z^n - S_1 z^{n-1}+\ldots+(-1)^n S_n = \begin{cases} \prod_{j=1}^{m}(z^2+\omega_j^2) & \text{for } n \text{ even} \\ z\prod_{j=1}^{m}(z^2+\omega_j^2) & \text{for } n \text{ odd,} \end{cases} \tag{14}$$

where S_p are elementary symmetrical functions of the ω_j. If all the eigenvalues of $\mathbf{\Omega}$ are distinct, the minimal polynomial equals the characteristic polynomial and then S_p is the sum of the principal minors of order p of the determinant of $\mathbf{\Omega}$. Obviously $S_p = 0$ if p is odd. Noting that

$$\lambda_q = \frac{1}{q!}\left(\frac{\mathrm{d}^q f(z)}{\mathrm{d}z^q}\right)_{z=0}, \qquad (q = 0, 1, \ldots, n-1) \tag{15}$$

the coefficients λ_q can be found by straightforward calculation from (13), and substitution in (11) leads to the desired finite power series. If $\mathbf{f}(\mathbf{\Omega})$ is supposed to be a real operator, the result is

for $n = 2m$:

$$\mathbf{f}(\mathbf{\Omega}) = \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(\mathrm{Re}\, f(\mathrm{i}\omega_j)\mathbf{I}+\mathrm{Im}\, f(\mathrm{i}\omega_j)\frac{\mathbf{\Omega}}{\omega_j}\right), \tag{16a}$$

for $n = 2m+1$:

$$\mathbf{f}(\mathbf{\Omega}) = f(0)\mathbf{I}+\sum_{j=1}^{m} \mathbf{A}_j \cdot \left(\mathrm{Im}\, f(\mathrm{i}\omega_j)\frac{\mathbf{\Omega}}{\omega_j}+(f(0)-\mathrm{Re}\, f(\mathrm{i}\omega_j))\frac{\mathbf{\Omega}^2}{\omega_j^2}\right), \tag{16b}$$

where in both expressions

$$\mathbf{A}_j = \sum_{l=0}^{m-1} \frac{(-1)^l k_{2l}(\mathrm{i}\omega_j)}{\omega_j^2 \Pi_{p=1,p\neq j}^{m}(\omega_p^2-\omega_j^2)}\frac{\mathbf{\Omega}^{2l}}{\omega_j^{2l}}. \tag{17}$$

By $k_{2l}(z)$ is denoted the polynomial obtained by omitting all the terms containing z raised to a power greater than $2l$ in $k(z)$ (n even) or $k(z)/z$ (n odd), and $k_{2l}(\mathrm{i}\omega_j)$ in (17) is equal to $\Sigma_{r=0}^{l}(-1)^r S_{2(m-r)}\omega_j^{2r}$.

In particular the closed form of the matrix exponential is found to be

for $n = 2m$:

$$\exp(\mathbf{\Omega}t) = \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(\cos \omega_j t \mathbf{I} + \frac{\sin \omega_j t}{\omega_j} \mathbf{\Omega}\right), \tag{18a}$$

for $n = 2m+1$:

$$\exp(\mathbf{\Omega}t) = \mathbf{I} + \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(\frac{\sin \omega_j t}{\omega_j} \mathbf{\Omega} + \frac{1-\cos \omega_j t}{\omega_j^2} \mathbf{\Omega}^2\right). \tag{18b}$$

This last equation is a generalization of a formula deduced by Chang and Audeh (1970) in the case of three dimensions, using direct recursion relations for $\mathbf{\Omega}$.

After substitution of t by $-t$ in (18) we get the inverse matrices

for $n = 2m$:

$$\exp(-\mathbf{\Omega}t) = \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(\cos \omega_j t \mathbf{I} - \frac{\sin \omega_j t}{\omega_j} \mathbf{\Omega}\right), \tag{19a}$$

for $n = 2m+1$:

$$\exp(-\mathbf{\Omega}t) = \mathbf{I} + \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(-\frac{\sin \omega_j t}{\omega_j} \mathbf{\Omega} + \frac{1-\cos \omega_j t}{\omega_j^2} \mathbf{\Omega}^2\right). \tag{19b}$$

In the same way the matrix $\mathbf{P} = \exp(-\mathbf{\Omega}t) \cdot \int_0^t \mathrm{d}t \int_0^t \exp(\mathbf{\Omega}t)\,\mathrm{d}t$ occurring in (10) can be written as a finite power series in $\mathbf{\Omega}$. Putting in (16)

$$f(\mathrm{i}\omega_j) = \exp(-\omega_j t) \int_0^t \mathrm{d}t \int_0^t \exp(\mathrm{i}\omega_j t)\,\mathrm{d}t = \frac{1-\exp(-\mathrm{i}\omega_j t) - \mathrm{i}\omega_j t \exp(-\mathrm{i}\omega_j t)}{(\mathrm{i}\omega_j)^2} \tag{20a}$$

$$f(0) = \tfrac{1}{2}t^2, \tag{20b}$$

there results

for $n = 2m$:

$$\mathbf{P} = \sum_{j=1}^{m} \frac{\mathbf{A}_j}{\omega_j^2} \cdot \left([\omega_j t \sin \omega_j t - (1-\cos \omega_j t)]\mathbf{I} + (\omega_j t \cos \omega_j t - \sin \omega_j t)\frac{\mathbf{\Omega}}{\omega_j}\right), \tag{21a}$$

for $n = 2m+1$:

$$\mathbf{P} = \frac{t^2}{2}\mathbf{I} + \sum_{j=1}^{m} \frac{\mathbf{A}_j}{\omega_j^2} \cdot \left([\omega_j t \cos \omega_j t - \sin \omega_j t]\frac{\mathbf{\Omega}}{\omega_j} + (\tfrac{1}{2}\omega_j^2 t^2 + 1 - \cos \omega_j t - \omega_j t \sin \omega_j t)\frac{\mathbf{\Omega}^2}{\omega_j^2}\right), \tag{21b}$$

with $\mathbf{A}_j$ given again by (17).

Substitution of (19) and (21) into (10) yields the general solution for $\boldsymbol{q}$ in a closed form. Leaving the initial conditions very general: when $t = 0$: $\boldsymbol{q}(0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}_0 = \boldsymbol{v}_0$, we find

for $n = 2m$:

$$\boldsymbol{q} = \sum_{j=1}^{m} \frac{\mathbf{A}_j}{\omega_j^2} \cdot \left([\omega_j t \sin \omega_j t - (1 - \cos \omega_j t)]\boldsymbol{F} + (\omega_j t \cos \omega_j t - \sin \omega_j t) \frac{\boldsymbol{\Omega} \cdot \boldsymbol{F}}{\omega_j} \right)$$
$$+ \sum_{j=1}^{m} \mathbf{A}_j \cdot \left((\omega_j t \sin \omega_j t + \cos \omega_j t)\boldsymbol{q}_0 + (\omega_j t \cos \omega_j t - \sin \omega_j t) \frac{\boldsymbol{\Omega} \cdot \boldsymbol{q}_0}{\omega_j} \right)$$
$$+ \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(t \cos \omega_j t \boldsymbol{v}_0 - t \sin \omega_j t \frac{\boldsymbol{\Omega} \cdot \boldsymbol{v}_0}{\omega_j} \right), \tag{22a}$$

for $n = 2m+1$:

$$\boldsymbol{q} = \frac{t^2}{2}\boldsymbol{F} + \sum_{j=1}^{m} \frac{t^2}{2} \mathbf{A}_j \cdot \frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{F}}{\omega_j^2} + \boldsymbol{q}_0 + t\boldsymbol{v}_0$$
$$+ \sum_{j=1}^{m} \frac{\mathbf{A}_j}{\omega_j^2} \cdot \left((\omega_j t \cos \omega_j t - \sin \omega_j t) \frac{\boldsymbol{\Omega} \cdot \boldsymbol{F}}{\omega_j} + (1 - \cos \omega_j t - \omega_j t \sin \omega_j t) \frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{F}}{\omega_j^2} \right)$$
$$+ \sum_{j=1}^{m} \mathbf{A}_j \cdot \left((\omega_j t \cos \omega_j t - \sin \omega_j t) \frac{\boldsymbol{\Omega} \cdot \boldsymbol{q}_0}{\omega_j} \right.$$
$$\left. + (1 - \cos \omega_j t - \omega_j t \sin \omega_j t) \frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{q}_0}{\omega_j^2} \right)$$
$$+ \sum_{j=1}^{m} \mathbf{A}_j \cdot \left(-t \sin \omega_j t \frac{\boldsymbol{\Omega} \cdot \boldsymbol{v}_0}{\omega_j} + t(1 - \cos \omega_j t) \frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{v}_0}{\omega_j^2} \right). \tag{22b}$$

3. Application to particle motion in a rotating frame

The equation of motion describing particle motion in a three-dimensional space with respect to a rotating coordinate system having the same origin as the fixed frame, is known to be (Goldstein 1950)

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_{\mathrm{a}} + \boldsymbol{F}_{\mathrm{t}} + \boldsymbol{F}_{\mathrm{c}}, \tag{23}$$

with $\boldsymbol{F}_{\mathrm{a}}$: the absolute force; $\boldsymbol{F}_{\mathrm{t}} = -m[\dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})]$: the force of transport $\boldsymbol{F}_{\mathrm{c}} = -2m(\boldsymbol{\omega} \times \dot{\boldsymbol{r}})$: the Coriolis force. To expose the announced connection with (4), equation (23) is written as

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \frac{\boldsymbol{F}_{\mathrm{a}}}{m}. \tag{24}$$

If a skew-symmetric matrix $\boldsymbol{\Omega}$ is associated with $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ according to

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix},$$

each cross product $\boldsymbol{\omega} \times \boldsymbol{x}$ can be replaced by an inner product $\boldsymbol{\Omega} \cdot \boldsymbol{x}$. In this way (24) is reduced to (4).

The equation is of particular interest for describing the motion of a heavy particle near the surface of the earth; then $\boldsymbol{\Omega}$ and $\boldsymbol{F}_a/m = \boldsymbol{g}$ may be treated as constant and (24) becomes:

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega} \cdot \dot{\boldsymbol{r}} + \boldsymbol{\Omega}^2 \cdot \boldsymbol{r} = \boldsymbol{g}. \tag{25}$$

The exact solution of this equation has been obtained by von Eberhard (1930) and Leroy (1971) after projection on the axes of a well chosen coordinate system. A frame-independent vector solution has been obtained by Verheest and Leroy (1973) using a representation in circularly polarized coordinates.

The method of the matrix exponential, discussed in this paper, enables us to write down this solution immediately. The eigenvalues of $\boldsymbol{\Omega}$ are 0 and $\pm i\omega$, with $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$; as $\mathbf{A} = \mathbf{I}$ for $N = n = 3$, we find for the matrix exponential from (18*b*)

$$\exp(\boldsymbol{\Omega}t) = \mathbf{I} + \frac{\sin \omega t}{\omega}\boldsymbol{\Omega} + \frac{1 - \cos \omega t}{\omega^2}\boldsymbol{\Omega}^2,$$

and for the general solution from (22*b*)

$$\begin{aligned} \boldsymbol{r} = {} & \frac{t^2}{2}\boldsymbol{g} + \frac{t^2\boldsymbol{\Omega}^2 \cdot \boldsymbol{g}}{2\omega^2} + \boldsymbol{r}_0 + t\boldsymbol{v}_0 \\ & + (\omega t \cos \omega t - \sin \omega t)\frac{\boldsymbol{\Omega} \cdot \boldsymbol{g}}{\omega^3} + (1 - \cos \omega t - \omega t \sin \omega t)\frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{g}}{\omega^4} \\ & + (\omega t \cos \omega t - \sin \omega t)\frac{\boldsymbol{\Omega} \cdot \boldsymbol{r}_0}{\omega} + (1 - \cos \omega t - \omega t \sin \omega t)\frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{r}_0}{\omega^2} \\ & - t \sin \omega t \frac{\boldsymbol{\Omega} \cdot \boldsymbol{v}_0}{\omega} + t(1 - \cos \omega t)\frac{\boldsymbol{\Omega}^2 \cdot \boldsymbol{v}_0}{\omega^2}, \end{aligned} \tag{26}$$

which after we return to a notation with cross products and choose the initial condition $\boldsymbol{r}_0 = 0$, leads us to the solution found by Verheest and Leroy (1973).

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